Bugs and Slips in Bracket Expansions: A Calculator Comparison.

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Abstract

Studies have shown that many students make systematic errors when expanding linear algebraic brackets. Some of these errors could be classified as ***bugs*** *(of a procedural nature involving an incorrect routine), but most are* ***slips*** *(of a careless nature). Many slips are caused by difficulties experienced within working memory. This study investigated whether the use of a calculator could reduce the load on working memory and, subsequently reduce the number of slips made. A comparison was made between students, who had use of calculators, and students who had no calculators on bracket expansion tasks. An analysis of errors found that the use of a calculator did not reduce the number of errors or alter the types of errors made.*

The analysis of student errors in mathematics has long been valued by educators (see Resnick and Ford, Ch.4, 1981; Maurer, 1987; Ashlock,1986). Error patterns can reflect common misconceptions and wrongly applied strategies. Awareness of specific errors can help mathematics teachers select appropriate remedial actions, as errors can indicate both knowledge and lack of knowledge. From this perspective, **systematic** errors are particularly useful. For example, if a student continually multiplies two negatives together to make a negative, the error is easily identifiable, and appropriate action may be taken. Similarly, if a student regularly makes co-interior angles between parallel lines equal, the teacher can easily categorise the problem and act accordingly. Brown and Burton (1978) identified a class of systematic errors in arithmetic which they called **BUGS**. These types of errors were procedural and often featured an incorrect routine in an otherwise correct method. For example, Brown and Burton found that many students would routinely have trouble "borrowing from zero" on the subtraction problem 705-9, and finishing with the answer 796. In comparison to bugs, Brown and Burton generally categorised errors which were not part of systematic procedures as careless errors, often referred to as **SLIPS**. Whereas knowledge of bugs has been considered valuable, slips have proved less useful because of their more random nature. There are many reasons why a student will make a careless error, and therefore these types of error are harder to classify and act on.

However, slips by their very nature commonly involve forgetfulness. A student may forget some part of the problem, incorrectly remember data presented in the problem, or even recall information from long term memory wrongly, such as the multiplication tables. In other words, slips are very much a memory phenomenon. In particular, many slips are caused by a breakdown in short-term memory, or as it is usually called these days: working memory. For the purpose of this article, working memory (see Baddeley, 1986, for a more extensive definition) is considered to be a system which holds and processes information; as Schoenfeld (1992) remarked "where thinking gets done" (p. 350).

Recent research by Ayres and Sweller (1990), and Ayres (1993, 1995), has suggested that many types of mathematical problems can produce systematic slips. These are errors which are not bugs, but nevertheless, appear systematically. In the Ayres & Sweller (1990), and Ayres (1993) studies on 2 stage geometry problems which required the calculation of a subgoal and goal only, many subjects would make significantly more errors on the subgoal stage compared with the goal stage, even though the tasks were equally demanding. For example, a subject might successfully apply parallel lines properties when they appeared in the goal, but make errors when the same applications were needed at the subgoal stage. The authors argued that at the subgoal stage cognitive load (see Chandler and Sweller, 1991, for further discussion on Cognitine Load Theory) was higher than the goal stage. Subsequently, more demands were made on working memory at the subgoal stage which forced more errors to occur.

In a different domain, Ayres (1995), found that bracket expansions also produced similar error profiles. In these experiments students were asked to expand a set of brackets isomorphic to the problem shown below (referred to as **Problem-1**):

$$-3(-4x+5) - 4(3-3x).$$

To multiply out the brackets successfully, four operations have to take place:

1. $-3 * -4x$ 2. $-3 * 5$ 3. $-4 * 3$
4. $-4 * -3x$

Results from this study found that many students made significantly more errors completing Operations 2 and 4, compared with Operations 1 and 3 respectively. Students also made more errors expanding the second bracket compared with the first bracket. In particular, errors were caused by incorrect sign-multiplication, rather than numerical calculations. A student would often multiply two negatives correctly when they appeared in Operations 1 and 3, but incorrectly when they appeared in Operations 2 and 4. Verbal protocols provided evidence that most errors could not be classified as bugs. Although one bug (applying the negative before the bracket in Operation 3, but ignoring it in Operation 4) could be identified consistently for some students, it was not systematically applied by many. Most evidence suggested that the errors were slips. Recall experiments in this study also supported this argument and suggested a strong link between the demands made on working memory and the location of errors. It was hypothesised from these experiments that cognitive load was greater during Operations 2 and 4 than 1 and 3 respectively, and also for the second bracket compared with the first. Whereas students with poor working memories (measured by recall) in this domain exhibited the described error patterns, students with stronger working memories did not.

It should be noted that all students who participated in the Ayres (1995) study were not allowed to use calculators. In bracket expansions, knowledge of bracket concepts, algebraic symbols, sign multiplications and general arithmetic must be accessed from long term memory, and manipulated in working memory. Some students may experience high cognitive loads if they are not allowed use of a calculator. Having a calculator available for numerical and sign multiplications may reduce errors. The following study investigated the effects of using a calculator during bracket expansions. In particular, error profiles for incorrect sign-multiplications was investigated.

Method

Subjects

Thirty nine students in Year 8 (20 from the top class, 19 from the 5th-streamed class) from a high school in Western Sydney participated. All students had some experience in expanding brackets.

Materials and procedure

Each student received a set of eight problems isomorphic to problem 1 (shown above) on a single sheet of paper with sufficient space after each problem to complete their answers. The set of problems were couterbalanced to ensure that the four types of sign multiplication , (+ * +, + * -, - * +, - * -), were equally distributed over the set in each operational position. This design created an unbiased instrument for investigating error distributions. A group of students from both classes were chosen at random and allowed to use calculators; the remaining students were not allowed to use calculators. This selection process grouped 17 students (7 from class 5, 10 from class 1) into a Calculator group and 22 students (10 from class 5, 12 from class 1) into a Non-calculator group. Students in the calculator group were encouraged verbally at the start of the exercise to use their calculators as much as possible. Enough time was given for students to complete the task. All students were instructed to expand the brackets only but not group terms.

Results and discussion

Twenty eight subjects made errors, 6 made no errors and 5 subjects (all from class 5) were rejected. Subjects (2 from the Calculator group and 3 from the Non-calculator group) were rejected because they could not complete all problems in the set. This exclusion was necessary in order to preserve the counterbalancing of the four types of sign multiplication. An incomplete set of answers would bias the results.

Incorrect multiplication of signs contributed 80% of all errors made. The remaining 20% consisted of numerical errors, 45% of which were made by one subject. Numerical errors consisted of incorrect multiplication (16%), excluding a premultiplier (32%), adding or subtracting instead of multiplying (38%), and miscellaneous (14%). As this experiment was designed to investigate sign multiplications, numerical errors were not included in the following analysis. Mean number of errors made by each group are shown in Table 1.

Table 1

	Class 1	Class 5	Combined
Calculator Group	M=2.4	M=8.6	M=4.5
	SD=2.8	SD=2.7	SD=4.0
Non-Calculator Group	M=2.4	M=7.6	M=4.3
	SD=2.7	SD=4.7	SD=4.4

As subjects were chosen from two different ability classes, a 2 x 2 ANOVA was performed. For the Calculator-use main effect , there was no significant difference ($F = 0.69$, $MSE = 1.95$, $p > 0.05$) indicating that the use of calculators did not reduce the number of errors made. As expected there was a significant difference between classes ($F = 20.4$, $MSE = 245$, $p < 0.01$), but no calculator x class interaction ($F = 0.17$, $MSE = 2.1$, $p > 0.05$).

All errors were categorised according to the operation location in which they occurred. The distribution of errors for each group is shown in Table 2.

Table 2

Calculator Group

	Class 1	Class 2	Class 3
Operation 1	12%	4%	9%
Operation 2	23%	38%	29%
Operation 3	23%	20%	22%
Operation 4	42%	38%	40%

Non-Calculator Group

Operation 1	13%	14%	13%
Operation 2	24%	22%	23%
Operation 3	14%	14%	14%
Operation 4	49%	50%	50%

These data enabled individual error profiles to be collected for each subject. In the previous study by Ayres (1995), more errors were made during Operation 2 compared with Operation 1, and Operation 4 compared with Operation 3, as well as Bracket 2 compared with Bracket 1. These comparisons were tested on this present data using one-tailed Wilcoxon match-paired tests (see Table 3). The number of subjects employed in this experiment was not large enough to analyse this information by each class group, and therefore only Calculator and Non-calculator group comparisons were made.

Table 3

Combined Classes

Calculator Group		
Op. 2 v Op. 1	$T(8)=0$,	$p < 0.05$
Op.4 v Op. 2	$T(8)=4$,	$p < 0.05$
Br. 2 v Br. 1	$T(11)=15.5$,	$p > 0.05$
Non-Calculator Group		
Op. 2 v Op. 1	$T (12)=21$,	$p >0.05$
Op.4 v Op. 2	$T (11)=0$,	$p < 0.05$
Br. 2 v Br. 1	$T (12)=16$,	$p < 0.05$.

There were significantly more errors made on Operation 4 compared with Operation 3 for both the Calculator group and the Non-calculator group. For Operations 2 and 1, only the Calculator Group proved significant, although the Non-calculator group approached significance. For bracket comparisons, the Non-calculator group made significantly more errors in the second bracket compared with the first, while the Calculator group just missed significance at the 95% level. Overall, it can be concluded that use of the calculator has not decreased the number of errors made by subjects (see Table 1) or significantly altered the error distributions over the four operations, compared with subjects who did not have access to a calculator (see Table 3).

Analysis of individual errors revealed that few (7%) sign-multiplication errors were made by premultiplying by a "+". The majority (93%) of errors were made by premultiplying by a "-". The problem set was designed such that premultiplication by a negative occurred four times in each operational position. Analysis of subject errors was conducted on these positions to investigate the existence of bugs. The most frequent bug found in the Ayres (1995) study was that subjects would not include the minus before the bracket during the fourth operation. If a subject consistently used this bug, a total of four errors in this position would be made. Four subjects from the Calculator group and three subjects from the Non-calculator group exhibited this behaviour. However, verbal protocols were not collected in this experiment and, therefore, it was not possible to ascertain beyond doubt the existence of bugs. One other subject made four errors when premultiplying by a negative in the second operational position. This was also potentially a bug. Apart from these cases there was no evidence to suggest that subjects from either group consistently applied an incorrect procedure. In fact, error profiles suggested that most students made slips.

Bugs, if they were employed, were used in a partial or inconsistent manner.

General discussion

The results from this experiment have shown that the use of the calculator has not reduced either the number of errors , or the types of errors made on the given task. Subjects, whether they used a calculator or not., still made more errors during Operations 4 and 2, compared with Operations 3 and 1, respectively. More errors were also made during the expansion of Bracket 2 compared with Bracket 1. It should be noted that the availability of a calculator does not necessarily imply it will be used. The experiment was not designed to observe employment of this aid. Similarly, there was no measure of the subjects' competence with calculators. A future study could investigate these factors. Consequently, the author wishes to acknowledge that these results should not be generalised too much at this point. It may be that other conditions will produce a different result. However, the findings in this study have shown again that students make systematic errors when expanding brackets. Many of these errors will be slips rather than bugs. Careless errors, by their nature, are often dependent upon working memory. The demands of the problem may produce an error, just as much as lack of knowledge. Mathematics teachers, on discovering systematic errors, should proceed with some caution: the bug may be a systematic slip.

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